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Comparative Statement of New Business for the years 1849 *and* 1850.

	1849.			1850.		
	No. of Policies on Lives.	Sum Assured.	Premium.	No. of Policies on Lives.	Sum Assured.	Premium.
		£. s. d.	£. s. d.		£. s. d.	£. s. d.
Whole life . .	361	156,462 10 0	4,985 13 1	403	184,241 1 0	6,063 12 7
For terms . .	20	11,533 6 8	382 4 8	27	10,785 13 4	232 1 4
Special contract.	19	10,094 0 0	283 18 0	19	9,210 0 0	187 4 10
	400	178,089 16 8	5,651 15 9	449	204,236 14 4	6,482 18 9

Annuities.

1849.		1850.	
No. of Policies.	Amount Sunk.	No. of Policies.	Amount Sunk.
	£. s. d.		£. s. d.
68	19,177 2 5	72	29,083 15 3

THE INSTITUTE OF ACTUARIES.

The establishment of the "*Association of Managers of the Scottish Life Assurance Offices*" has, we believe we may say with perfect truth, been productive of the greatest advantage to the public, the offices, and the managers themselves. By its means the latter have been enabled to attain to a consistency of principles, and an uniformity of practice, tending powerfully to promote the welfare of their Companies, and to secure for them the respect and confidence of the public. Relieved from the weight of an unpleasing and unnecessary responsibility, each member of the Association, informed by the knowledge and strengthened by the opinions of the rest, is in a position to exercise in his own particular sphere of action that wholesome influence and control so essential to the proper conduct and government of the complicated machinery which it is his peculiar province to regulate, and which, directed by unskilful hands, must so soon relapse into inextricable disorder. Bound together by their common interests, and stimulated only by a generous and liberal rivalry, they form a phalanx sufficiently strong to resist by its moral force the encroachments of irregular enterprises, and to suppress such as are of a more doubtful character; and thus the great and important objects which these institutions have in view are carried out in the sister kingdom in the most economical, advantageous, and effective manner.

It must, we think, be admitted, that had such an association been formed at an early period amongst the managers or actuaries of the London offices, many of the existing evils connected with the business of Life Assurance in England would not have arisen. There appears to be no sufficient reason why such a measure should not have been attended with the same results here as in Scotland. Organization and unity of action, once established amongst the Companies first formed, would have tended to retard that rapid increase in their numbers which is operating now so disadvantageously. The managers themselves would have been able from time to time to devise the means of checking that unwholesome growth. Better informed generally, and supported by the co-operation of all, each would have been enabled to inspire greater confidence, and to maintain a more influential position. The necessity for retaining, in every case, the services of one so endowed, would have been more fully recognised and steadily kept in view, and greater difficulty would have attended the establishment of a Company where the aid of such an officer was not obtained—whilst on the other hand, persons of such character and standing would not have lent themselves to the furtherance of undertakings of a doubtful or illegitimate description.

The apparently obvious remedy which these observations point out, viz., the immediate resort to a precisely similar association, is unfortunately not now practicable, as regards the whole body; the number of offices is so great, that an assembly which included all their managers would be too large for the convenient discussion of the questions likely to come before them—whilst the body would not be content to conform in such matters to the decisions of a few selected from amongst them. In the lapse of time too, many if not all are committed to certain modes of procedure, and cannot now retrace their steps; moreover, their very want of union has not only created a great diversity of practice, but has very materially led to the placing many in a position deprived of that controlling or guiding power without which their knowledge and experience are all but unavailing; and many would thus be quite unable to carry out measures previously determined upon, let them be ever so well calculated for the general advantage.

We believe that it was such considerations as these—felt, though not expressed—as led to the *Institute of Actuaries* taking the peculiar character and features which it did, immediately upon its creation. It was felt that nothing could be more futile than to meet for the purpose of agreeing to that which it was known could not be agreed to, and of resolving what it would not, in many cases, be possible to execute. Minor associations, it was true, could be formed, of a more or less exclusive character; and, as regarded the managers of offices of similar calibre and analogous interests, could no doubt be formed, especially for official purposes, with some prospect of advantage: but these alone would have rather tended to perpetuate the antagonisms which discordant interests had already created, and could never have brought about any improvement in the condition of the general body, or have effected any change which would be calculated for its permanent advancement.

A little consideration sufficed to show that the evils which called for remedy had taken deeper root, and required more radical reformation; a more widely extended knowledge of principles was seen to be necessary, and a better acquaintance with the methods of applying them, before so many different views were likely to be reconciled; and when this should be achieved, there would still be wanting the power required to give effect to those more carefully elaborated opinions. Thus it became evident that the whole body must acquire a greater public consideration, before it could be useful in repressing the dangerous abuses so prevalent on every side, and in securing for the system that consistency and purity in its operation, without which it was likely to become a public pest rather than a public blessing. Hence the Institute, immediately upon its development, assumed almost involuntarily a collegiate or educational character, and while it sought to comprehend within its bounds all the existing members of the profession, and to afford them the means of intercommunion and discussion, strove to foster and encourage the learning peculiar to their pursuits, to improve the existing modes of practice, and to fix its mark on such as might thereafter distinguish themselves for the attainment of the one, or the exhibition of greater skill in the other. It was felt that a certain power was wanting, and it was seen that the acknowledged possession of the required learning and experience would be likely to confer that power.

With what success this measure has been hitherto attended, we will not at this stage of it stop to inquire; but from what we have already seen, we feel confident in predicting that concurrently with the rapid growth and vigorous maturity of the Institute, a better order of things will speedily develope itself; an improved tone will take the place of the very indifferent one heretofore perceptible; more just, and liberal, and accurate ideas on the several points connected with the common pursuit will be prevalent, and greater confidence exhibited in the enunciation of them; differences in practice will vanish with differences of opinion. Armed with the moral power thus gradually acquired, and, it may be, shielded by legal protection, the profession will ultimately assume in England as well as in Scotland that position, the attaining of which we have all along insisted upon, as essential to the due protection of private as well as public interests; and which, enabling it to give the proper direction to honest and legitimate enterprises, will place in its hands the means of exposing and extinguishing such as are of an unsound or delusive description, and of putting a stop to the mischievous expedients to which their projectors may, from time to time, think proper to resort. X.

PROCEEDINGS OF THE INSTITUTE.

First Sessional Meeting, 1850-51; *25th November*, 1850.

MR. JELLICOE, V.P., in the Chair.

The minutes of the last Sessional Meeting having been read and confirmed, the Vice-President announced that the following contributions had been made to the Library:—

	Presented by
Milne on Annuities, 2 vols.	Mr. Cleghorn.
Gardiner's Logarithms, 1742	Mr. Wagner.
Addison's Works, 4 vols.	Mr. P. Hardy, V.P.
Life of Lord Chancellor Bacon	Mr. P. Hardy, V.P.
Boyle's Experiments on Air	Do.
Brown's Mathematical Tables	Do.
Report on Church Leases	Do.

	Presented by
Cleland's Glasgow Statistical Tables	Mr. P. Hardy, V.P.
Report on Drunkenness	Do.
Geographical Dictionary, 2 vols.	Do.
The Justice, &c. of an Income or Property Tax	Do.
Nicholson's Dictionary of Chemistry	Do.
Population Abstracts of Great Britain, 1811	Do.
Transactions of the Statistical Society, 1837	Do.
Sturmin's Course of Mathematics, 3 vols.	Do.
Walter's Mathematical Dictionary	Do.
Bonnycastle's Trigonometry, 2nd edition	Mr. Jenkin Jones.
British Almanack and Companion for 1850	Do.
Papers of the Central Society of Education	Do.
Minutes of the Committee of Council on Education, 1839-40	Do.
Do. Do. 1840-1	Do.
Do. Do. 1842-3	Do.
Do. Do. 1844	Do.
Francœur's Pure Mathematics, 2 vols.	Do.
Hirsch's Geometry	Do.
Report on Local Taxation	Do.
Ludlam's Rudiments of Mathematics	Do.
Metropolitan Sanitary Commission, 1848	Do.
Report on the Continuance of the Poor Law Commissions	Do.
Poor Law Commissioners' 2nd Annual Report, 1836	Do.
Do. 3rd Do. 1837	Do.
Do. 4th Do. 1838	Do.
Do. 5th Do. 1839	Do.
Do. 6th Do. 1840	Do.
Do. 7th Do. 1841	Do.
Do. 9th Do. 1843	Do.
Three Reports on the Irish Poor Law	Do.
Extracts from the Three Reports on the Irish Poor Law	Do.
Registrar-General's 1st Annual Report 1839	Do.
Do. 3rd Do. 1841	Do.
Do. 4th Do. 1842	Do.
Sanitary Inquiry, England, 1842	Do.
Do. Scotland, 1842	Do.
Working Man's Year Book, 1835	Do.
Essay on Annuities, by a Military Fund Director, with an Appendix	Mr. Noble.
Reports on the Mining Districts	Mr. Colvin.
Saunderson's Algebra	Do.
Simpson's Algebra	Do.
Simpson's Fluxions, 2 vols.	Do.
Whiston's Elementa Geometriæ	Do.
Ansell on Friendly Societies	Mr. Higham.
Corbaux on Population, Vitality, and Mortality	Do.
De Morgan on Probabilities	Mr. Winser.

It was resolved unanimously that the thanks of the meeting be given to the above-mentioned gentlemen, severally, for their liberal donations to the Library.

The meeting was also informed that the Council had made the following purchases for the Library:—

Alison's History of Europe, 7th edition, 20 vols.
Cooper's Surgical Dictionary.
Cuvier's Animal Kingdom.
Encyclopædia Metropolitana, 30 vols.
Fariday's Electricity, 2 vols.
Hooper's Medical Dictionary.
Kirby and Spence's Entomology, 2 vols.

Macaulay's England, 2 vols.
Macaulay's Historial Essays.
McCullock's Geographical Dictionary, 2 vols.
McCullock's Dictionary of Commerce.
Murray's Encyclopædia of Geography.
Phillips' Geology, 2 vols.
System of Universal Geography.
Tredgold on the Steam Engine.
Turner's Chemistry.
Walker on the Electric Telegraph.

Mr. Peter Hardy, V.P., then read a paper "On the values of Annuities, which are to pay certain given rates of interest on the purchase-money during the whole term of their continuance, and to replace their original values on their expiration, at certain other given rates."

WE devote, with much pleasure, a few pages of the *Magazine* to the following paper furnished by Mr. Porter. The manner in which he has treated the several questions, appears to us to be well worthy of the consideration of the junior members of the profession. The solutions to the questions proposed at the second day's examination will, we hope, appear in our next number; and the future candidate for the Institute's certificate then having one complete type of the process before him, will, we trust, derive from it useful hints for his own guidance.

Solutions of the Questions proposed to Candidates for the "Certificate of Competency," at the First Examination, held on the 10*th and* 11*th June,* 1850; *with an Investigation of the principal Formulæ required in their Solution. By* H. W. PORTER, Esq., *of the Alliance Life Assurance Company.*

Note.—In presenting to the readers of the *Assurance Magazine* the following solutions, it is necessary to state, that as the space allotted for them in its pages is but limited, it has been necessary to abridge the answers to the questions as much as possible.

Plain and simple solutions of each question have been aimed at throughout; and it will be borne in mind that they are intended, not for the instruction of the senior members of the actuarial profession, but for the information and guidance of junior students.

Where the investigation of some of the formulæ required in the solution of the questions has been considered necessary, the matter will be found enclosed within brackets.

The questions proposed at the *vivâ voce* examination are divided into two sections, the one theoretical and the other practical.—H. W. P.

Examination Paper.—Monday, 10*th June,* 1850.

"*Levia quidem hæc, et parvi forte, si per se spectentur, momenti. Sed ex elementis constant, ex principiis oriuntur, omnia. Et ex judicii consuetudine in rebus minutis adhibita, pendet sæpissime etiam in maximis vera atque accurata Scientia.*" — Clarke's Homer's Iliad: Preface.

1. Show how a decimal quantity can be expressed in the form of a vulgar fraction.

Decimal fractions are such as have for their denominators 10 or some power of 10; and these, reduced to their lowest terms, give the equivalent vulgar fractions.

$$\text{Ex. } {\cdot}245 = \frac{245}{1000} = \frac{49}{200}.$$

2. Find the fractional value of the recurring decimal ·27272727.

$$\text{Let } 27{\cdot}272727 = \text{S}.$$

Multiplying both sides of the equation by 100, we have

$$27{\cdot}272727 = 100 \text{ S}.$$

Then by subtracting the former equation from the latter,

$$27 = 99 \text{ S}$$

$$\therefore \text{S} = \frac{27}{99} = \frac{3}{11}.$$

[Note.—Recurring decimals are made up of quantities in Geometrical Progression, where $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. is the Common Ratio, according as one, two, or three figures recur. Thus ·27272727 is equivalent to the Geometrical Series,

$$\frac{27}{10^2} + \frac{27}{10^4} + \frac{27}{10^6} + \text{\&c. . . . . . ad infinitum,}$$

and may be summed by applying the formula $\text{S} = \frac{a}{1-r}$. (*vide* note to question No. 4, second day's paper),

$$\text{where } a = \frac{27}{10^2}, \text{ and } r = \frac{1}{10^2}$$

$$\therefore \text{S} = \frac{\frac{27}{10^2}}{1 - \frac{1}{10^2}} = \frac{27}{99} = \frac{3}{11} \text{ as before.]}$$

3. Find by logarithms the cube root of a number whose square root is 86·37129.

Since the square root of the number = 86·37129 ∴ the number itself = $(86{\cdot}37129)^2$, and its cube root = $(86{\cdot}37129)^{\frac{2}{3}}$, to find which

$$\begin{array}{rrl}
\lambda\ 86{\cdot}37129 = & 1{\cdot}9363679 & \\
 & 10 & \\
 & 46 & \\
\hline
 & 1{\cdot}9636936 & \\
 & 2 & \\
\hline
3\ ) & 3{\cdot}87273872 & = \lambda \text{ number required.} \\
\hline
 & 1{\cdot}2909129 & = \lambda \text{ cube root;}
\end{array}$$

and the natural number equivalent to this is 19·53934.

4. Find the amount of £375, in 50 years, at £5 per cent. per annum, compound interest; and the present value of £500, to be received at the end of 14 years at 3 per cent. per annum, compound interest.

(1) Let s = the amount.
p = the principal.
n = the number of years.
i = the interest of £1 for 1 year.

Then $s=p(1+i)^n$ and $\lambda s=\lambda p+n\,.\,\lambda(1+i)$

$p=£375, \quad n=50, \quad i={\cdot}05$

$$\begin{array}{rr} \lambda 375=2{\cdot}5740313 & \lambda 1{\cdot}05={\cdot}0211893 \\ 1{\cdot}0594650 & 50 \\ \hline 3{\cdot}6334963 \ \ N=4300{\cdot}275 & 1{\cdot}059465 \end{array}$$

£4,300. 5*s*. 6*d*.

(2) $p=\dfrac{s}{(1+i)^n}=s\times(1+i)^{-n}$ and $\lambda p=\lambda s-n\lambda\,.\,(1+i)$

$s=£500, \quad n=14, \qquad i={\cdot}03$

$$\begin{array}{rr} \lambda 500=2{\cdot}6989700 & \lambda 1{\cdot}03={\cdot}0128372 \\ {\cdot}1797208 & 2 \\ \hline 2{\cdot}5192492 \ \ N=330{\cdot}5592 & {\cdot}0256744 \\ & 7 \\ \hline & {\cdot}1797208 \end{array}$$

£330. 11*s*. 2*d*.

5. Solve the general quadratic equation $px^2-qx+r=o$, and show that the sum of the roots is equal to $\frac{q}{p}$, and their product to $\frac{r}{p}$.

(1) Multiplying by $4p$ the equation becomes

$$4p^2x^2-4pqx=-4pr.$$

Adding to each side of the equation q^2, we have

$$4p^2x^2-4pqx+q^2=q^2-4pr.$$

Extracting the square root,

$$2px-q=\pm\sqrt{q^2-4pr}$$

$$\therefore\ x=\pm\frac{\sqrt{q^2-4pr}+q}{2p}.$$

(2) The roots are $+\dfrac{\sqrt{q^2-4pr}+q}{2p}$ and $-\dfrac{\sqrt{q^2-4pr}+q}{2p}$

$\therefore$ by addition of these quantities we have $\frac{q}{p}$.

and by multiplication,

$$\frac{q+\sqrt{q^2-4pr}}{2p}\times\frac{q-\sqrt{q^2-4pr}}{2p}=\frac{4pr}{4p^2}=\frac{r}{p}.$$

6. Solve the equation $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{1-x^2}$.

It is convenient to change the form of this expression to that of fractional indices.

$$(1+x)^{\frac{2}{m}}-(1-x)^{\frac{2}{m}}=(1-x^2)^{\frac{1}{m}}$$

$$(1+x)^{\frac{2}{m}}-(1-x)^{\frac{2}{m}}=(1+x)^{\frac{1}{m}}\,(1-x)^{\frac{1}{m}}.$$

Dividing by $(1-x)^{\frac{2}{m}}$

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}}-1=\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}};$$

or,

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}}-\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}=1.$$

Now, completing the square, by adding to each side the square of half the coefficient of the second term:

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}}-\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}+\frac{1}{4}=1+\frac{1}{4}\text{ or }\frac{5}{4};$$

and

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}-\frac{1}{2}=\pm\frac{\sqrt{5}}{2}$$

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}=\frac{1\pm\sqrt{5}}{2}.$$

Raising both sides to the mth power,

$$\frac{1+x}{1-x}=\left(\frac{1\pm\sqrt{5}}{2}\right)^m$$

$\therefore$ Componendo et dividendo,

$$\frac{1+x+(1-x)}{1+x-(1-x)}=\frac{(1\pm\sqrt{5})^m+2^m}{(1\pm\sqrt{5})^m-2^m}$$

$$\therefore\quad \frac{1}{x}=\frac{(1\pm\sqrt{5})^m+2^m}{(1\pm\sqrt{5})^m-2^m}$$

$$\therefore\quad x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}.$$

7. Find the sum of the series 1, 3, 6, 10, &c. to n terms (the second differences being supposed constant).

If a, b, c, d, &c. be the term of a series whose first, second, third, &c. differences are expressed by $\delta 1$, $\delta 2$, $\delta 3$, &c., then the sum of n terms of the series is

$$na+n\,.\,\frac{n-1}{2}\,.\,\delta_1+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\delta_2+\&\text{c.}\ldots\ldots(\textit{vide}\text{ note.})$$

(In the given series the second differences being supposed constant, it is not necessary to write down any more terms.)

$$\begin{array}{ccccccc} 1, & & 3, & & 6, & & 10, \\ & 2 & & 3 & & 4 & \\ & & 1 & & 1 & & \\ & & & 0 & & & \end{array}$$

Here $a=1$, $\delta_3=2$, $\delta_1=1$, $\delta_2=o$

$$\therefore \text{ the sum is } n+n\,.\,\frac{n-1}{2}\,.\,2+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}$$

$$=n+n\,.\,\overline{n-1}+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.$$

$$=n^2+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.$$

[Note.—To investigate formulæ for the summation of series by the differential method.

Let a, b, c, d, e, f, g, &c., be a series following a regular law :

$$\text{And let }\left.\begin{aligned} b-a &= a' \\ c-b &= b' \\ d-c &= c' \\ e-d &= d' \\ f-e &= e' \end{aligned}\right\}\ \text{Then } a,\ b,\ c,\ d,\ \&\text{c. are called the 1st order of differences.}$$

$$\left.\begin{aligned} b'-a' &= a'' \\ c'-b' &= b'' \\ d'-c' &= c'' \\ e'-d' &= d'' \\ f'-e' &= e'' \end{aligned}\right\}\ a'',\ b'',\ \&\text{c.}\qquad \text{2nd order of differences.}$$

$$\left.\begin{aligned} b''-a'' &= a''' \\ c''-b'' &= b''' \\ d''-c'' &= c''' \\ e''-d'' &= d''' \\ f''-e'' &= e''' \end{aligned}\right\}\ a''',\ b''',\ \&\text{c.}\qquad \text{3rd order of differences.}$$

and so on.

1st proposition.
To find the nth order of differences, *i.e.* to find a general expression.

$$\begin{aligned} \text{From above } a'' &= b'-a' = c-b-(b-a) = c-2b+a \\ b'' &= c'-b' = d-c-(c-b) = d-2c+b \\ c'' &= d'-c' = e-d-(d-c) = e-2d+c \\ d'' &= e'-d' = f-e-(e-d) = f-2e+d \\ \text{also } a''' &= b''-a'' = d-2c+b-(c-2b+a) = d-3c+3b-a \\ b''' &= c''-b'' = e-2d+c-(d-2c+b) = e-3d+3c-b \\ c''' &= d''-c'' = f-2e\,.\,d-(e-2d+c) = f-3e+3d-c \end{aligned}$$

and similarly,

$$\begin{aligned} a'''' = b'''-a''' &= e-3d+3c-b-d+3c-3b+a \\ &= e-4d+6c-4b+a \end{aligned}$$

Hence we conclude generally, that the law holds, which is, that the coefficients follow the same law as the coefficient of $(a-x)^n$. (*Vide* question 13, Binomial theorem.)

Now, writing the above in reversed order, we have,

$$\begin{array}{lcl} a' & = & -(a-b) \\ a'' & = & a-2b+c \\ a''' & = & -(a-3b+3c-d) \\ a'''' & = & a-4b+6c-4d+e \\ \&c. & & \&c. \end{array}$$

$$\therefore a^n = \pm(a-nb+n\,.\,\overline{\frac{n-1}{2}}\,.\,c-n\,.\,\overline{\frac{n-1}{2}}\,.\,\overline{\frac{n-2}{3}}\,.\,d+\&c.\;.\;.\;.$$

$+$ or $-$ according as n is even or odd, and upon this depends the system of interpolation.

It is important to observe here that the coefficients of the

$$\begin{array}{lcl} \text{2nd term are as if from} & & (1+x)^1 \\ \text{3rd} & ,, & (1+x)^2 \\ \text{4th} & ,, & (1+x)^3 \\ \quad\&c. & \&c. & \\ \therefore & & \\ (n+1)\text{th} & ,, & (1+x)^n \\ \text{and} & & \\ n\text{th} & ,, & (1+x)^{n-1}. \end{array}$$

2nd proposition.

To find the $(n+1)$th term of the above series.

The series is,

$$a,\ b,\ c,\ d,\ e,\ f,\ \&c.$$

$$\begin{array}{lll} \text{and the 2nd term} & b=a+a' & \\ \text{3rd} \;,, & c=b+b'=a+\;a'+a' & +\;a''=a+2a'+a'' \\ \text{4th} \;,, & d=c+c'=(a+2a'+a'') & +\;(b'+b'') \\ & \quad=(a+2a'+a'') & +\;(a'+a''+a''+a''') \\ & \quad=(a+2a'+a'') & +\;(a'+2a''+a''') \\ & \quad=\;a+3a'+3a'' & +\;a''', \end{array}$$

and so on.

$$\therefore (n+1)\text{th term} = a+na'+n\,.\,\frac{n-1}{2}\,.\,a''+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,a'''+\&c.$$

3rd proposition.

To express the sum of the series.

The $(n+1)$th term of the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.$$

is the sum of n terms of the series

$$a+b+c+d+\&c.$$

and we have the series

$$0,\quad a,\quad a+b,\quad a+b+c,\quad a+b+c+d,\quad \&c.\ .$$

$$a-0,\ a+b-a,\ a+b+c-a-b,\ a+b+c+d-a-b-c,\ \&c.\ .\ .\ .\ \delta^0=a$$

or

$$\begin{array}{lllll} a, & b, & c, & d & \\ b-a & c-b & d-c & & \ldots\ \delta'=b-a \\ c-b-b+a & d-c-c+b & & & \end{array}$$

or

$$\begin{array}{lll} c-2b+a & d-2c+b & \ldots\ \delta''=c-2b+a \\ d-2c+b-c+2b-a & & \end{array}$$

or

$$d-3c+3b-a \qquad \ldots\ \delta'''=d-3c+3b-a$$

$\therefore$ we have

$$a+b+c+d+\ .\ .\ .\ .\ \text{to } n \text{ terms.}$$

$$=0+n\delta 0+\frac{n\,.\,\overline{n-1}}{2}\,.\,\delta 1+n\,.\,\frac{\overline{n-1}}{2}\,.\,\frac{\overline{n-2}}{3}\,.\,\delta 2+\,.\,.\,.\,.\,.$$

$$=na+\frac{n\,.\,\overline{n-1}}{2}\,.\,\delta 1+n\,.\,\frac{\overline{n-1}}{2}\,.\,\frac{\overline{n-2}}{3}\delta 2+\,.\,.\,.$$

δ in this proposition being put generally for a' b' c' &c. . .]

8. Find the sum of n terms of the series $1+2x+3x^2+4x^3+$ &c.

Let S=required sum.

$$\therefore\ S=1+2x+3x^2+4x^3+\,.\,.\,.\,.\,.\,nx^{n-1}.$$

Multiplying by x, we have

$$xS=x+2x^2+3x^3+\,.\,.\,.\,.\,.\,+(n-1)\,x^{n-1}+nx^n.$$

Subtracting the latter equation from the former—

$$S-xS=1+x+x^2+x^3+\,.\,.\,.\,.\,.\,x^{n-1}-nx^n,$$

which may be also expressed in the form

$$S\,(1-x)=\frac{1-x^n}{1-x}-nx^n$$

$$\therefore\ S=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}.$$

9. Explain what is meant by the logarithm of a number, in respect to a given base, and express algebraically the relation between a number and its logarithm.

The logarithm of any number to a given *base* is the power to which the base must be raised to give that number.

Example.—If 2 is the base, then $2^5=32$ $\therefore$ log. 32 is 5 to the base 2. 10, however, is the base used in all calculations.

If x is the logarithm of N to the base a, then the equation connecting a, N, and x, is $N=a^x$.

10. Explain the use of the table of proportionate parts in using tables of logarithms, and give an example of its application.

First, to form a table of proportionate parts :

Let log. 14430=159266, writing only the mantissæ or decimal
log. 14420=158965 part.

Difference 10 301

Here we see that if the 5th place be taken as the units place, the difference of 10 in the number gives a difference of 301 in the logarithm.

The question then arises, if 10 be added to the number 14420 for an addition of 301 to its logarithm, what must be added if 1, 2, 3, 4, &c. be respectively added?

Hence we have the following proportion:—

$$\begin{array}{rcccccccc} \text{As } 10 & : & 1 & :: & 301 & : & 30{\cdot}1 \\ 10 & : & 2 & :: & 301 & : & 60{\cdot}2 \\ 10 & : & 3 & :: & 301 & : & 90{\cdot}3 \\ 10 & : & 4 & :: & 301 & : & 120{\cdot}4 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 10 & : & 9 & :: & 301 & : & 270{\cdot}9 \end{array}$$

TABLE.

	Differences.	
N	300	299
1	30	30
2	60	60
3	90	90
4	120	120
5	150	150
6	180	179
7	210	209
8	240	239
9	270	269

The same principle applies to the 6th place, except that 301 then corresponds to a difference of 100 instead of 10, and therefore the difference in the table must be divided by 10, which in practice is done by moving the difference one unit's place to the right.

Example from above table of differences.

Find the logarithm of 1442·139, of which the first 5 figures only are found at sight in ordinary tables of logarithms.

$$\begin{array}{ll} \text{Log. } 1442{\cdot}1 & = 3{\cdot}1589954 \\ \left.\begin{array}{l}\text{Prop. parts from above table,} \\ \quad\text{difference being } 299 \ldots\ldots\end{array}\right\} & \begin{array}{l} 3 \ldots\ldots \ \ 90 \\ 9 \ldots\ldots \ \ 269 \end{array} \\ \hline & 3{\cdot}15900709 = \text{log. } 1442{\cdot}139 \end{array}$$

11. Find the number of permutations which can be effected with n distinct symbols.

Let there be n letters,

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad . \quad . \quad . \quad a_n.$$

If we take away a_1, then $n-1$ letters only remain.

$$a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad . \quad . \quad . \quad a_n.$$

Again, placing a_1 before each,

$$a_1a_2 \quad a_1a_3 \quad a_1a_4 \quad . \quad . \quad . \quad . \quad a_1a_n,$$

which gives $n-1$ permutations of n letters two and two in which a_1 stands first.

Adopting the same course with a_2,

$$\begin{array}{cccccccc} a_2a_1 & a_2a_3 & a_2a_4 & . & . & . & . & a_2a_n \\ a_3a_1 & a_3a_2 & a_3a_4 & . & . & . & . & a_3a_n \\ \vdots & \vdots & \vdots & & & & & \vdots \\ a_na_1 & a_na_2 & a_na_3 & . & . & . & . & a_na_{n-1}, \end{array}$$

each line containing $n-1$ combinations, and the number of the lines being n,

$\therefore$ whole number $= n\,.\,\overline{n-1}\,. = \text{P}2$, or the permutations of n quantities taken two and two together.

Further,

Taking away a_1, and permuting the remaining $\overline{n-1}$ letters two and two, we have by the first case $\overline{n-1}\,.\,n-2$ permutations.

$$a_2a_3 \qquad a_3a_4 \qquad . \quad . \quad . \quad . \quad . \qquad a_{n-1}a_n$$

Replacing a_1,

$$a_1a_2a_3 \qquad a_1a_3a_4 \qquad . \quad . \quad . \quad . \quad . \qquad a_1a_{n-1}a_n;$$

here, then, are $\overline{n-1}\,.\,\overline{n-2}$ permutations 3 and 3 together, where a_1 stands first, and similarly $\overline{n-1}\,.\,\overline{n-2}$ permutations where a_2, $a_3 \ldots a_n$ stand first;

$\therefore$ whole number of permutations taken 3 and 3 together is $n\,.\,\overline{n-1}\,.\,\overline{n-2}$.

Further,

It seems probable, therefore, that the permutations of n letters taken r and r together,

$$=n\,.\,\overline{n-1}\,.\,\overline{n-2} \quad . \quad . \quad . \quad . \quad \overline{n-r-1}.$$

Let us for the present assume that this is true.

Let there be n letters as before,

$$a_1 \ a_2 \ a_3 \ a_4 \ . \ . \ . \ . \ . \ . \ . \ a_n,$$

and let a_1 be taken away, and the remaining $\overline{n-1}$. letters be permuted r and r together.

Then the number is $\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}\,.\ -\ -\ -\ \overline{n-r}$.

Now if the formula be true for r and r it is equally true for $r+1$ and $r+1$. But it has been proved to be true for 2 and 2 and 3 and 3. $\therefore$ it is true for 4, and if for 4 for 5, &c., and $\therefore$ it is true generally.

$$P2=n\,.\,\overline{n-1}$$

$$P3=n\,.\,\overline{n-1}\,.\,\overline{n-2}$$

$$Pn=n\,.\,\overline{n-1}\,.\,\overline{n-2} \quad - \quad - \quad - \quad \overline{n-r-1}$$

$$Pn=n\,.\,\overline{n-1}\,.\,\overline{n-2} \quad - \quad - \quad - \quad 3\,.\,2\,.\,1$$

12. Prove that the total number of different combinations of n things taken 1, 2, 3 . . . n at a time is 2^n-1.

$(a+x)^n=$ (expanding by the Binomial theorem.)

$$a^n+na^{n-1}x+\frac{n\,.\,n-1}{1\,.\,2}a^{n-2}x^2+n\,.\,\frac{n-1\,.\,n-2}{1\,.\,2\,.\,3}a^{n-3}x^3 \quad . \quad . \quad .$$

$$. \quad . \quad . \quad . \quad . \quad \frac{n\,.\,n-1}{1\,.\,2}a^{n-2}x^{n-2}+nax^{n-1}+x^n.$$

Let $a=1$, $x=1$.

$$\therefore \ 2^n=1+n+\frac{n\,.\,n-1}{1\,.\,2}+\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}+\&c. \ . \ . \ . \ . \ . \ .$$

$$. \quad . \quad . \quad . \quad . \quad \frac{n\,.\,n-1}{1\,.\,2}+n+1.$$

$$\text{And } 2^n-1=n+n\,.\,\frac{n-1}{1\,.\,2}+n\,.\,\frac{n-1\,.\,n-2}{1\,.\,2\,.\,3}+\ .\ .\ \frac{n\,.\,n-1}{1\,.\,2}+n+1$$

$$=C_1+C_2+C_3+\ .\ .\ C_{n-2}+C_{n-1}+C_n.$$

C_1, C_2, C_3, &c., representing the total number of combinations of N things taken 1 together, 2 together, 3 together, &c. . . n together.

[Note.—In this proof it has been assumed that the number of combinations of n things taken r and r together = the number taken $n—r$ together, which may be proved as follows :—

If we take n quantities $a_1, a_2, a_3, \ldots\ldots\ldots a_n$, and form any one combination of r of these quantities, those which remain will form a corresponding and supplementary combination of $n—r$ quantities; it follows ∴ that no combination can exist without its supplement, and consequently they must be equal in number.]

13. State and prove the Binomial theorem for the case where the index is a positive integer.

The Binomial theorem shows how to raise a Binomial to any power by an easier and shorter method than by repeated multiplication.

There are several proofs of the Binomial theorem; almost every writer on algebra gives a different proof. Perhaps the following is the most simple.

Let $(x+a)$ be a Binomial.

Its nth power, or $(x+a)^n$, is

$$x^n+nax^{n-1}+n\,.\,\frac{n-1}{2}\,.\,a^2x^{n-2}+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,a^3x^{n-3}+\,\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\quad \frac{n\,.\,\overline{n-1}}{2}\,.\,a^{n-2}x^{n-2}+na^{n-1}x+a^n.$$

Where the index of x beginning from n is diminished by unity, and the index of a beginning from o is increased by unity, also the coefficient of each term is found by multiplying the coefficient of the preceding term by the index of x in that term, and dividing by the index of a increased by unity.

To investigate the theorem when the index is a positive integer:

By actual multiplication of n quantities $(x+a)$, $(x+b)$, $(x+c)$, &c., together, it appears that

$$(x+a)\,(a+b) = x^2+(a+b)\,x+ab$$
$$(x+a)\,(x+b)\,(x+c) = x^3+(a+b+c)\,x^2+(ab+ac+bc)\,x+abc$$
$$(x+a)\,(x+b)\,(x+c)\,(x+d) = x^4+(a+b+c+d)\,x^3+(ab+ac+bc+ad+bd+dc).$$
$$\times\,x^2+(abc+acd+adc+bcd)\,x+abcd$$

and the same law of the continued product is observed to hold, whatever be the number of Binomial factors $(x+a)$, $(x+b)$, $(x+c)$, &c., actually multiplied together; viz., that it is composed of a descending series of the powers of x, the index of the highest being the number of the factors, and the other indices decreasing by unity in each successive term—also the coefficient of the first term is 1, of the second term the quantities a, b, c, &c.; of the third, the sum of the product of every two; the fourth, the sum of the product of every three, and so on; of the last, the product of all the n quantities, a, b, c, d, &c.

Suppose, then, that this law holds for n Binomial factors

$(x+a), (x+b), (x+c)\ldots.(x+k)$ so that
$(x+a).\,(x+b).\,(x+c)\ldots.(x+k)=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots.K$
Where $A=a+b+c+\ldots\ldots\ldots\ldots k$
$B=ab+ac+bc+\ldots\ldots\ldots bk$
$C=abc+acd+\ldots\ldots\ldots\ldots bck$
&c.
$K=abcd\ldots\ldots\ldots\ldots k.$

Now if we introduce a new factor $(x+l)$ we have

$$(x+a)(x+b)(x+c)\ldots(x+k)(x+l)=x^{n+1}+(A+l)x^n+(B+lA)x^{n-1}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots Kl$$

$$\begin{aligned}\text{Hence } A+l&=a+b+c+d+\ldots\ldots\ldots k+l\\ B+lA&=ab+ac+bc+\ldots\ldots\ldots al+bl+kl\\ &\vdots\\ Kl&=abcd\ldots\ldots\ldots\ldots\ldots kl\end{aligned}$$

So that we see if the law holds when n Binomial factors are multiplied together, the same law *is proved* to hold for $n+1$ factors.

But it has been shown to hold for four factors $\therefore$ it is true for five, and if for five then also for six, and so on.

Now let $a=b=c=d=$&c.

Then $A=a+a+a+a+\ldots\ldots$ to n terms $=na$

$$B=a^2+a^2+a^2+\ldots\ldots\ldots \left.\begin{array}{r}\text{to as many terms}\\ \text{as there are combinations}\\ \text{of } n \text{ things taken 2 and 2}\end{array}\right\}=n\,.\,\frac{n-1}{2}\,.\,a^2$$

$$C=a^3+a^3+a^3+\ldots\ldots\ldots \left.\begin{array}{r}\text{to as many terms}\\ \text{as there are combinations}\\ \text{of } n \text{ things taken 3 and 3}\end{array}\right\}=n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,a^3$$

$$\vdots$$

$$K=a\,.\,a\,.\,a\,.\,a\ldots\ldots\ldots\ldots\ldots \text{ to } n \text{ factors } = a^n$$

Also $(x+a)(x+b)(x+c)\ldots(x+k)$ becomes $(x+a)^n$ $\therefore$ $(x+a)^n$

$$=x^n+nax^{n-1}+n\,.\,\frac{n-1}{2}\,.\,a^2x^{n-2}+\ldots\ldots\ldots\ldots a^n$$

14. A, B, C, D, E, &c., being any series whose fourth differences are zero, express the value of E in terms of the preceding quantities.

The nth term of the above series is

$$a+(n-1)\delta1+(n-1)\frac{n-2}{2}\,.\,\delta2+(n-1\,\frac{n-2}{2}\,.\,\frac{n-3}{3}\,.\,\delta3+\text{\&c.}$$

(*vide* note to question 7.)

$$\begin{array}{ccccc} A & B & C & D & E\end{array}$$

$$\begin{array}{lll} B-A, & C-B, & D-C, \quad E-D = \delta_1 \text{ if a descending series}\\ C-B-B+A, & D-C-C+B, & E-D-D+C=\delta2\\ =C-2B+A, & D-2C+B, & E-2D+C\\ D-2C+B-C+2B-A, & & E-2D+C-D+2C-B=\delta3\\ = \quad D-3C+3B-A, & & E-3D+3C-B\\ & & \qquad 0 = \delta4\end{array}$$

The 4th differences being zero.

Then if E be the nth term, we have

$$A+(n-1)\,.\,\delta1+(n-1)\,.\,\frac{n-2}{2}\,\delta\,+(n-1)\,.\,\frac{n-2}{2}\,.\,\frac{n-3}{3}\,\delta3+0$$

$$\text{or}\quad A+(n-1)\,.\,(B-A)+(n-1)\,.\,\frac{n-2}{2}\,.\,(C-2B+A)+$$
$$+\,(n-1)\,.\,\frac{n-2}{2}\,.\,\frac{n-3}{3}\,.\,(D-3C+3B-A)$$

Now writing for n its numerical value, 5, E being the fifth term, we have

$$A+4B-4A+6C-12B+6A+4D-12C+12B-4A$$
$$=-A+4B-6C+4D.$$

15. The probability of the happening of an event in one trial being $\frac{a^1}{a}$, show the respective probabilities of the happening of the same event once only, and once at least in three trials.

(1.) The probability of the event happening at any given trial is $\frac{a^1}{a}$, and of its failing $1-\frac{a^1}{a}=\frac{a-a^1}{a}$

$\therefore$ the chance of its happening once and failing twice is

$$\frac{a^1}{a}\times\frac{a-a^1}{a}\times\frac{a-a^1}{a},\ \text{or}\ \frac{a^1\,.\,(a-a^1)^2}{a^3}.$$

Now there are three ways in which it may happen once and fail twice, *i.e.*, it may happen, fail, fail,
or fail, happen, fail,
or fail, fail, happen.

Hence the probability required $=\frac{3a^1\,(a-a^1)^2}{a^3}$.

The condition of its happening once at least will be satisfied in any of these cases, *i.e.*,

1st. If it happen once and fail twice,
or 2nd. If it happen twice and fail once,
or 3rd. If it happen all three times.

The probability of the first is, as before $=\frac{3a^1\,(a-a^1)^2}{a^3}$.

Similarly the probability of the second $=\frac{3(a^1)^2\,(a-a^1)}{a^3}$.

And the probability of the third...... $=\frac{(a^1)^3}{a^3}$.

$\therefore$ the probability of the event happening once at least is the sum of these, viz.:

$$\frac{(a^1)^3+3(a^1)^2\,(a-a^1)+3a^1\,(a-a^1)^2}{a^3}.$$

Aliter (2).

Unless the event happen once at least, it will fail in all three trials, the probability of which is $\left(\frac{a-a^1}{a}\right)^3$ and the probability of its happening once at least is $\therefore\ 1-\left(\frac{a-a^1}{a}\right)^3$.

16. What is the probability, according to De Moivre's hypothesis, that one or the other of three persons aged 20 will be alive at the end of 10 years?

De Moivre's hypothesis was suggested by Halley's Table, constructed from observations made in Breslau: he considered that of 86 persons born, one dies every year till all are extinct. The expectation of life by this hypothesis is always = half the number living at each age.

Upon this hypothesis, therefore, the number living at the age of 20 will be 66, and at 30 the number living will be 56. The chance, therefore, of one of the three living 10 years is $\frac{56}{66}$, and of one of the three dying within that period is $\frac{10}{66}=\frac{5}{33}$, and of all the three dying within 10 years, $\left(\frac{5}{33}\right)^3$.

The probability ∴ that one or other will be living at the end of 10 years is $1-\left(\frac{5}{33}\right)^3$.

17. Find on the above-mentioned hypothesis, the present value of £250, to be received in the event of two persons, aged 30 and 50, both being alive at the end of 14 years, money being supposed to double itself in that time.

The number living on this hypothesis at the age of 30, is as before, 56, and at the age of 44 is 42; the probability, therefore, of a person aged 30 living 14 years, is $\frac{42}{56}=\frac{3}{4}$.

Similarly, the probability of a person aged 50 living 14 years, is $\frac{22}{36}=\frac{11}{18}$

∴ the probability of both being then alive, is $\frac{3}{4}\times\frac{11}{18}=$

$\frac{1}{4}\times\frac{11}{6}=\frac{11}{24}=$ ·45833, which decimal multiplied into £250, gives £114·583, the present value of £250 on the above contingency, irrespective of the interest of money.

This result multiplied by ·5, money being supposed to double itself in the given time, gives £57. 5*s.* 10*d.*, the required result.

Vivâ voce Examination.

Section I.

No. I. What is the third power of $-\sqrt{-1}$?

$$(-\sqrt{-1})^3=\sqrt{-1}.$$

2. What is the numerical value of the base *e* of the Naperian logarithms?

The value of *e*, which is a constant quantity, is 2·7182818 ....

[Note.—Napier, the inventor of logarithms, adopted this base because logarithms to it are more easily calculated than those to any other base.

All logarithm tables are first calculated to base e, and afterwards converted into Briggs' or common logarithms by multiplying by $\frac{1}{l_{e10}}$, the numerical value of which is ·434294819 . . . .

This is called the *modulus* of the system whose base is 10.

There are several formulæ for the calculation of logarithms.

Those generally made use of are the following:

$$l_e x = 2 \cdot \left\{ \frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots \right\}$$

and having found $l_e x$,

$$l_e (1+x) = l_e x + 2 \left\{ \frac{1}{2x+1} + \frac{1}{3} \cdot \frac{1}{(2x+1)^3} + \frac{1}{5} \cdot \frac{1}{(2x+1)^5} + \dots \right\}$$

This being a very rapidly converging series, the terms set down are probably sufficient for any practical purpose.

The investigation of these formulæ will be found in any treatise on the Theory of Logarithms.]

3. Define harmonical progression.

The quantities A, B, C, D, E, are in harmonical progression, when

$A : C :: A - B : B - C$
$B : D :: B - C : C - D$
$C : E :: C - D : D - E$ &c.

Also the reciprocals of quantities in harmonical progression are in arithmetical progression.

By the definition, since

$A : C :: A - B : B - C$

$\therefore$ $AB - AC = AC - BC$, and dividing by ABC

$$\frac{AB}{ABC} - \frac{AC}{ABC} = \frac{AC}{ABC} - \frac{BC}{ABC}$$

or $\frac{1}{C} - \frac{1}{B} = \frac{1}{B} - \frac{1}{A}.$

Again, since

$B : D :: B - C : C - D$

$\therefore$ $BC - BD = BD - DC$, and dividing by BCD

$$\frac{BC}{BCD} - \frac{BD}{BCD} = \frac{BD}{BCD} - \frac{DC}{BCD}$$

or $\frac{1}{D} - \frac{1}{C} = \frac{1}{C} - \frac{1}{B}$

and $\frac{1}{C} - \frac{1}{B} = \frac{1}{B} - \frac{1}{A}$

$\therefore$ $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}$

have a common difference, and $\therefore$ they are in arithmetical progression.

(*To be continued in the next Number.*)